Applied Stochastic Processes (2018) - Final exam

Time: 3 hours, Total marks: 50

1. Consider the random walk S_n, $n \geq 0$ where $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for all $n \geq 1$. Here ξ_i, $i \geq 1$ are i.i.d. with $\mathbf{P}(\xi_1 = +1) = p$, $\mathbf{P}(\xi_1 = -1) = 1 - p$ and $1/2 < p < 1$. Let $\phi(x) = [(1-p)/p]^x$, $x \in \mathbb{Z}$.

   (a) Show that $\phi(S_n)$, $n \geq 0$ is a martingale with respect to $\mathcal{F}_n = \{S_1, S_2, \cdots S_n\}$. **[4 marks]**

   (b) Let $T_x = \inf\{n : S_n = x\}$. For $a < 0 < b$ integers, let $N = T_a \wedge T_b := \min(T_a, T_b)$. Show that $\mathbf{P}(N < \infty) = 1$. **[5 marks]**
   Hint: If the random walk takes $b - a$ consecutive $+1$ steps at any point of time, it will be out of the interval (a, b).

   (c) Show that $\phi(S_{N \wedge n})$, $n \geq 0$ is a martingale with respect to $\mathcal{F}_n = \{S_1, S_2, \cdots S_n\}$. **[3 marks]**

   (d) Show that $\mathbf{E}\phi(S_N) = 1$. **[4 marks]**

   (e) Using the above show that for $a < 0 < b$

   $$\mathbf{P}(T_a < T_b) = \frac{\phi(b) - 1}{\phi(b) - \phi(a)}. \qquad \textbf{[3 marks]}$$

   (f) Show that

   $$\mathbf{P}(T_a < \infty) = \left[\frac{1-p}{p}\right]^{-a}. \qquad \textbf{[2 marks]}$$

2. Let X_n, $n \geq 0$ be a branching process with $X_0 = 1$. Define the sequence $Y_r = X_{2018r}$, $r \geq 0$.

   (a) Show that Y_r, $r \geq 0$ is a branching process. **[4 marks]**

   (b) Let ϕ be the probability generating function of X_1. Find the probability generating function of Y_1. **[4 marks]**

3. Consider an infinitely many server queue (customers immediately find a server) with an exponential service time distribution with parameter μ. Suppose customers arrive in batches with the interarrival time following an exponential distribution with parameter λ. The number of arrivals in each batch is assumed to follow the geometric distribution with parameter ρ, $(0 < \rho < 1)$, i.e.

   $$\mathbf{P}(\text{no. of arrivals in a batch is } k) = \rho^{k-1}(1-\rho), \; k = 1, 2, \cdots.$$

   Formulate this process as a continuous time Markov chain and determine explicitly the Q-matrix (infinitesimal matrix) of the process. **[5 marks]**

4. Customers, with independent and identically distributed service time distribution H (and density h), arrive at a counter in the manner of a Poisson process with parameter λ. A customer who finds the server busy joins the queue with probability p, $(0 < p < 1)$. Derive the transition probabilities of the Markov chain embedded at the points of departure of customers. **[6 marks]**

5. Let me first recall the simple stochastic epidemic. We have a population of size $n+1$, with n susceptibles and 1 infected initially. Let X_t and Y_t denote the number of susceptibles and the number of infected, respectively, at time t. Given that at time t, $X_t = a$ and $Y_t = n+1-a$, we assume that during the time interval t to $t+\Delta t$, the probability of exactly one new infection is $\beta a(n+1-a)\Delta t + o(\Delta t)$ and that of no new infection is $1 - \beta a(n+1-a)\Delta t + o(\Delta t)$. There are no removals and the infected (also infectious) person remains in circulation foreover.

   Denote $p_r(t) = \mathbf{P}(X_t = r)$, $0 \leq r \leq n$ and let $F(x,t)$ denote its probability generating function:

   $$F(x,t) = \sum_{r=0}^{n} p_r(t) \cdot x^r$$

Show that $F(x,0) = x^n$ and

$$\frac{\partial F}{\partial t} = \beta(1-x)\left(n\frac{\partial F}{\partial x} - x\frac{\partial^2 F}{\partial x^2}\right) \qquad \textbf{[10 marks]}$$